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Best Approximate Solutions of Nonlinear Differential Equations

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1. INTRODUCTION

We consider best approximating the solution to

$$L[y] \equiv a(x)\,y'' + \sum_{i=1}^{n} F_i(x, y, y') + G(x, y, y') = h(x) \tag{1}$$

with initial conditions

$$y(0) = \beta_0\,, \qquad y'(0) = \beta_1 \tag{2}$$

by polynomials $p_k(x)$, in the sense that

$$\| L[y(x)] - L[p_k(x)] \| = \sup_{0 \leqslant x \leqslant c} | h(x) - L[p_k(x)] | \tag{3}$$

is a minimum.

Problems of this type have been considered by many, see [3, 4, 5, 6]. The results of this paper generalize results in [3, 4].

2. THE OPERATOR L

We assume that (1) satisfies the following conditions:

(i) The functions $F_i(x, y, y')$ and $G(x, y, y')$ are continuous at all points (x, y, y') of $[0, c]\,xR^2$, $1 \leqslant i \leqslant n$.

(ii) The functions $a(x)$ and $h(x)$ are continuous on $[0, c]$.

(iii) There exist functions $u(x)$ and $\phi(y, y')$ such that $u(x)$ is bounded and not zero on a countable set contained in $[0, c]$, $\phi(y, y')$ is continuous at all points (y, y') of R^2, $\phi(y, y') = 0$, if and only if, $y = 0$ or $y' = 0$, and for some α satisfying (v) below,

$$| G(x, y, y') | \geqslant r^{\alpha} | u(x) \phi(y/r , y'/r) |$$

for every $r \geqslant 1$.

(iv) There exist functions $M_i(x, y, y')$ which are continuous at all points (x, y, y') of $[0, c]\, xR^2$, and which for every $r \geqslant 1$, satisfy

$$| F_i(x, y, y')| \leqslant r^{\alpha_i} \left| M_i \left(x, \frac{y}{r}, \frac{y'}{r}\right) \right|$$

for some α_i , $1 \leqslant i \leqslant n$.

(v) The constant α in (iii) is such that $\alpha > \max(1, \alpha_1 ,..., \alpha_n)$.

Nonlinear operators satisfying conditions (i)–(v) are numerous. For example, let

$$L[y] \equiv y'' + f(x)[(y')^2/y^2 + 1] + g(x)\, y^{4/3}(y')^2\, e^{(1/y^2+1)},$$

where $f(x)$ and $g(x)$ are continuous on $[0, c]$, and $g(x)$ is not identically zero. Then

$$F_1(x, y, y') = f(x)[(y')^2/y^2 + 1],$$

$$G(x, y, y') = g(x)\, y^{4/3}(y')^2\, e^{1/y^2+1},$$

$$M_1(x, y, y') = F_1(x, y, y'), \qquad u(x) = g(x),$$

and

$$\phi(y, y') = y^{4/3}(y')^2.$$

The constant $\alpha_1 = 2$, and $2 < \alpha \leqslant 10/3$.

Also, the operators

$$L[y] \equiv y' + \sum_{i=1}^{n} a_i(x)\, y^i$$

and

$$L[y] \equiv y'' + b_1(x)\, y' + b_2(x)\, y + \sum_{i,j=0}^{m,n} a_{ij}(x)\, y^i(y')^j, \qquad i + j \geqslant 2$$

satisfy conditions (i)–(v), see [3, 4].

3. The Existence of Minimizing Polynomials

We now establish, for each $k \geqslant 1$, the existence of a polynomial of degree k which satisfies (2) and minimize, (3). The following Lemma will be useful in proving this result.

LEMMA. *Let* $S_1 = \{(c_0, c_1) \mid c_0^2 + c_1^2 \leqslant 1\}$, *and*

$$S_2 = \{(c_2, ..., c_k) \mid c_2^2 + \cdots + c_k^2 = 1\}.$$

If

$$p_k(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k,$$

then

$$\min_{S_1 \times S_2} \sup_{0 \leqslant x \leqslant c} | u(x) \phi(p_k, p_k')| = \sigma > 0.$$

Proof. Let $f(c_0, c_1, ..., c_k) = \sup_{0 \leqslant x \leqslant c} | u(x) \phi(p_k, p_k')|$. Then f is a continuous function on the compact set $S_1 \times S_2$. Suppose that $f(c_0^*, ..., c_k^*) = \sigma$ is the minimum value of f on $S_1 \times S_2$. If $\sigma = 0$, then from (iii),

$$\phi[p_k^*, (p_k^*)'] = 0 \text{ on a countable set,}$$

where

$$p_k^*(x) = c_0^* + c_1^* x + \cdots + c_k^* x^k.$$

Therefore, either $p_k^*(x) \equiv 0$ or $(p_k^*)'(x) \equiv 0$. This contradicts the linear independence of $\{1, x, x^2, ..., x^k\}$. Therefore, $\sigma > 0$.

THEOREM 1. *Let* (1) *satisfy conditions* (i)–(v). *Then there exists a linear combination* $p_k(x)$ *of* $\{1, x, x^2, ..., x^k\}$ *such that* (2) *is satisfied, and such that* (3) *is minimized among all sums of this type.*

Proof. It is clear that every $p_k(x)$ of the type considered here, must be of the form

$$p_k(x) = \beta_0 + \beta_1 x + c_2 x^2 + \cdots + c_k x^k.$$

According to Young's criterion ([2], p. 156), it is sufficient to show that

$$\| L[p_k(x)] \| \leqslant M \tag{4}$$

implies that $c_2^2 + c_3^2 + \cdots + c_k^2 \leqslant N$, where M and N are positive constants which are independent of the c_j's. Inequality (4) implies that

$$| G(x, p_k, p_k')| \leqslant M + | a(x) | \, | p_k'' | + \sum_{i=1}^{n} | F_i(x, p_k, p_k')|.$$

Suppose that $|a(x)| \leqslant A$ on $[0, c]$. Let $r^2 = c_2^2 + c_3^2 + \cdots + c_k^2$, and assume that $r^2 \geqslant \max(1, \beta_0^2 + \beta_1^2)$. Then (iii) and (iv) imply that

$$r^\alpha \,|\, u(x)| \,|\, \phi(p_k/r, p_k'/r)| \leqslant M + rA \,|\, p_k''/r \,| + \sum_{i=1}^{n} r^{\alpha_i} \,|\, M_i(x, p_k/r, p_k'/r)|.$$

Therefore, the Lemma and properties (iii) and (iv) imply that

$$r^\alpha \sigma \leqslant M + rAP + \sum_{i=1}^{n} r^{\alpha_i} K_i \,, \tag{5}$$

where $|\, p_k''(x)/r \,| \leqslant P$, and $|\, M_i(x, p_k/r, p_k'/r)| \leqslant K_i$, $0 \leqslant x \leqslant c$. We note that P and K_i are positive constants which are independent of the c_j's.

Because of (v), we may write α in the form $\alpha = \alpha^* + \epsilon$, where $\alpha^* \geqslant \max(1, \alpha_1, ..., \alpha_n)$, and where $\epsilon > 0$. Therefore, from (5) we have that

$$r^\epsilon \leqslant M/r^{\alpha^*} + AP/r^{\alpha^*-1} + \sum_{i=1}^{n} \frac{K_i}{r^{\alpha^*-\alpha_i}} \tag{6}$$

Since $r \geqslant 1$, inequality (6) implies that $r^\epsilon \leqslant N^*$, where N^* is independent of the c_j's. Therefore,

$$r^2 \leqslant \max[1, \beta_0^2 + \beta_1^2, (N^*)^{2/\epsilon}] = N,$$

completing the proof.

4. Convergence of Sequences of Approximating Polynomials

In order to arrive at the concluding theorem of this paper, we need an intermediate theorem. First, if $\bar{h} = (h_1, h_2)$, then $\|\bar{h}\|_a = |\,h_1\,| + |\,h_2\,|$.

Theorem 2. *Let* $\overline{W} = (w_1, w_2)$, $\bar{F}(x, \overline{W}) = [f_1(x, w_1, w_2), f_2(x, w_1, w_2)]$, *and let* $\{\bar{F}_k(x, \overline{W})\}$ *be a sequence of functions mapping* $[0, c] \times R^2$ *into* R^2 *such that*

$$\lim_{k\to\infty} \|\bar{F}_k(x, \overline{W}) - \bar{F}(x, \overline{W})\|_a = 0,$$

uniformly on every compact subset of $[0, c] \times R^2$. *Let* $\overline{W}_k(x)$ *be a solution of*

$$\overline{W}' = \bar{F}_k(x, \overline{W}), \qquad \overline{W}(0) = (\beta_0, \beta_1)$$

on $[0, c]$.

If $W(x)$ *is the unique solution of*

$$\overline{W}' = \bar{F}(x, \overline{W}), \qquad \overline{W}(0) = (\beta_0, \beta_1)$$

over [0, *c*], *then*

$$\lim_{k\to\infty} \| \overline{W}_k(x) - \overline{W}(x)\|_a = 0,$$

uniformly for $0 \leqslant x \leqslant c$.

The proof of this theorem is essentially that given in [1] and is omitted. We now state and prove the main theorem of this section.

THEOREM 3. *Suppose that there is a unique solution* $y(x)$ *over* [0, *c*], *to* (1) *and* (2), *with* $a(x) \equiv 1$. *If, for* $k = 1, 2,\ldots, p_k(x)$ *is a polynomial of degree* k *which satisfies* (2) *and minimizes* (3), *then* $p_k(x)$, $p_k'(x)$, *and* $p_k''(x)$ *converge uniformly throughout* [0, *c*] *to* $y(x)$, $y'(x)$, *and* $y''(x)$, *respectively, as* $k \to \infty$.

Proof. By an extension of the Weierstrass theorem, there exists a polynomial $q_k(x)$ of degree k, $k = 1, 2,\ldots,$ such that

$$q_k^{(j)}(0) = y^{(j)}(0), \qquad j = 0, 1, \text{ and such that}$$

$$\lim_{k\to\infty} q_k^{(j)}(x) = y^{(j)}(x), \qquad j = 0, 1, 2, \tag{7}$$

uniformly over [0, *c*]. Therefore,

$$\| L[y(x)] - L[p_k(x)]\| \leqslant \| L[y(x)] - L[q_k(x)]\|. \tag{8}$$

But

$$L[y(x)] - L[q_k(x)] = y'' - q_k'' + \sum_{i=1}^{n} [F_i(x, y, y') - F_i(x, q_k, q_k')] + G(x, y, y') - G(x, q_k, q_k').$$

Thus,

$$\| L[y(x)] - L[q_k(x)] \| \leqslant \| y'' - q_k'' \| + \sum_{i=1}^{n} \| F_i(x, y, y') - F_i(x, q_k, q_k')\| + \| G(x, y, y') - G(x, q_k, q_k')\|.$$

Therefore, (i) and (7) imply that

$$\lim_{k\to\infty} \| L[y(x)] - L[q_k(x)] \| = 0, \tag{9}$$

uniformly over [0, *c*].

Let

$$L[y(x)] - L[p_k(x)] = \epsilon_k(x). \tag{10}$$

Then (8) and (9) imply that

$$\lim_{k\to\infty} \epsilon_k(x) = 0, \tag{11}$$

uniformly over [0, c]. The system (1) and (2) may be written as

$$y'' = h(x) - \sum_{i=1}^{n} F_i(x, y, y') - G(x, y, y'),$$
$$y(0) = \beta_0, \qquad y'(0) = \beta_1.$$

This system is equivalent to the vector problem

$$\overline{W}' = \overline{F}(x, \overline{W}), \qquad \overline{W}(0) = (\beta_0, \beta_1), \tag{12}$$

where $\overline{W} = (w_1, w_2)$, and where

$$\overline{F}(x, \overline{W}) = \left[w_2, h(x) - \sum_{i=1}^{n} F_i(x, w_1, w_2) - G(x, w_1, w_2)\right].$$

From (10) we have that $p_k(x)$ is a solution to the differential system

$$y'' = h(x) - \sum_{i=1}^{n} F_i(x, y, y') - G(x, y, y') - \epsilon_k(x),$$
$$y(0) = \beta_0, \qquad y'(0) = \beta_1.$$

This system is equivalent to the vector problem

$$\overline{W}' = \overline{F}_k(x, \overline{W}), \qquad \overline{W}(0) = (\beta_0, \beta_1), \tag{13}$$

where

$$\overline{F}_k(x, \overline{W}) = \left[w_2, h(x) - \sum_{i=1}^{n} F_i(x, w_1, w_2) - G(x, w_1, w_2) - \epsilon_k(x)\right].$$

It follows from (11), (12), and (13) that

$$\lim_{k\to\infty} \|\overline{F}(x, \overline{W}) - \overline{F}_k(x, \overline{W})\|_a = 0.$$

Therefore, by Theorem 2, we conclude that

$$\lim_{k\to\infty} | y(x) - p_k(x)| + | y'(x) - p_k'(x)| = 0, \qquad 0 \leqslant x \leqslant c. \tag{14}$$

But

$$y'' - p_k'' = \epsilon_k(x) + \sum_{i=1}^{n} [F_i(x, p_k, p_k') - F_i(x, y, y')]$$
$$+ G(x, p_k, p_k') - G(x, y, y').$$

Therefore, (11) and (14) imply that

$$\lim_{k\to\infty} | y''(x) - p_k''(x)| = 0, \qquad 0 \leqslant x \leqslant c,$$

completing the proof.

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